Aerodynamics: Inviscid Aerodynamics & Boundary Layer Theory

1 Fluid Properties

1.1 Air Properties

- Ideal gas: $p/\rho = rT$; with $r = 287.14\,\mathrm{J/(kg\,K)}$
- Sea-level properties:
  - density: $\rho_0 = 1.225\,\mathrm{kg/m^3}$
  - pressure: $p_0 = 101325\,\mathrm{Pa}$
  - temperature: $T_0 = 288.15\,\mathrm{K}$
  - dynamic viscosity: $\mu_0 = 1.789 \cdot 10^{-5}\,\mathrm{Pa\,s}$
  - kinematic viscosity: $\nu_0 = \mu_0/\rho_0 = 1.4604 \cdot 10^{-5}\,\mathrm{m^2/s}$
  - const. press. spec. heat: $C_{p_0} = 1005\,\mathrm{J/(kg\,K)}$
  - const. vol. spec. heat: $C_{v_0} = C_{p_0} - r = 717.86\,\mathrm{J/(kg\,K)}$
  - adiabatic index: $\gamma_0 = C_{p_0}/C_{v_0} = 1.4$
  - thermal conductivity: $\lambda_{c_0} = 0.0251\,\mathrm{W/(m\,K)}$
  - Prandtl number: $Pr_0 = \dfrac{\mu_0\, C_{p_0}}{\lambda_{c_0}} = 0.7163$
- Standard Atmosphere relations:
  Troposphere ($h < 11km$, $T_h = -6.5 \cdot 10^{-3} K/m$):
  $\dfrac{T}{T_0} = \left(1 + \dfrac{T_h}{T_0} h\right)$, $\dfrac{p}{p_0} = \left(\dfrac{T}{T_0}\right)^{-\frac{g}{RT_h}}$, $\dfrac{\rho}{\rho_0} = \left(\dfrac{T}{T_0}\right)^{-\left(\frac{g}{RT_h}+1\right)}$
- Sutherland's law: $\dfrac{\mu}{\mu_{ref}} = \dfrac{T_{ref} + C}{T + C}\left(\dfrac{T}{T_{ref}}\right)^{3/2}$

  $\mu_{ref} = 1.711 \cdot 10^{-5}\,\mathrm{Pa\,s}$, $T_{ref} = 273.15\,\mathrm{K}$, $C = 110.4\,\mathrm{K}$

1.2 Water Properties

- At $T_0 = 293.15\,\mathrm{K}$
  - density: $\rho_0 = 998.3\,\mathrm{kg/m^3}$
  - dynamic viscosity: $\mu_0 = 1.002 \cdot 10^{-3}\,\mathrm{Pa\,s}$
  - kinematic viscosity: $\nu_0 = \mu_0/\rho_0 = 1.004 \cdot 10^{-6}\,\mathrm{m^2/s}$
  - scpecific heat capacity: $C_p = 4183\,\mathrm{J/(kg\,K)}$
  - thermal conductivity: $\lambda_{c_0} = 0.6\,\mathrm{W/(m\,K)}$
  - Prandtl number: $Pr_0 = \dfrac{\mu_0\, C_{p_0}}{\lambda_{c_0}} = 7.01$

2 General Equations of Fluid Motion

2.1 Differential formulation

- Continuity: $\dfrac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0$
- Momentum: $\rho \dfrac{\partial \vec{u}}{\partial t} + \rho\left(\vec{u} \cdot \nabla\right)\vec{u} = -\nabla p + \nabla \cdot \vec{\vec{\tau}} + \rho f_m$
- Energy:
  $\partial_t\left(\rho E\right) + \nabla \cdot \left(\rho E \vec{u} - \left(-p\vec{\vec{I}} + \vec{\vec{\tau}}\right) \cdot \vec{u} - \lambda_c \nabla T\right) = \rho \vec{f_m} \cdot \vec{u} + \rho \phi_\tau$
- Stress Tensor: $\vec{\vec{\tau}} = 2\mu\left[\dfrac{1}{2}\left(\nabla \vec{u} + \nabla \vec{u}^T\right) - \dfrac{1}{3}\left(\nabla \cdot \vec{u}\right)\vec{\vec{I}}\right]$
- Total Energy: $E = e + u^2/2$
- State Equations: $e = e(p, \rho)$ and $T = T(p, \rho)$

2.2 Integral formulation

- Continuity: $\dfrac{d}{dt}\iiint_{\mathcal{D}} \rho\, dV + \iint_{S} \rho\left(\vec{u} \cdot \vec{n}\right)\, dS = 0$
- Momentum: $\dfrac{d}{dt}\iiint_{\mathcal{D}} \rho \vec{u}\, dV + \iint_{S} \rho \vec{u}\,\left(\vec{u} \cdot \vec{n}\right)\, dS =$

  $= -\iint_{S} p\,\vec{n}\, dS + \iint_{S} \vec{\vec{\tau}} \cdot \vec{n}\, dS + \iiint_{\mathcal{D}} \rho f_m\, dV$

3 Inviscid Aerodynamics

3.1 Airfoil (2D)

- Circulation & Stokes Theorem: $\Gamma = \oint_C \vec{u} \cdot d\vec{l} = \iint_S \left(\nabla \times \vec{u}\right) \cdot \vec{n}\, d\sigma$
- Glauert method: $x = \dfrac{x_{TE} + x_{LE}}{2} + \dfrac{x_{TE} - x_{LE}}{2}\cos\left(\theta\right)$
  - lifting problem:

$$
\begin{aligned}
\frac{u^*(\theta)}{U_\infty} &= \frac{c_l\left(\theta\right)}{4} = A_0 tan\left(\frac{\theta}{2}\right) + \sum_{n=1}^{\infty} A_n \sin\left(n\theta\right) \\
\frac{w\left(\theta\right)}{U_\infty} &= \frac{dz_p}{dx} = -\alpha + \frac{dz_c}{dx} = -A_0 - \sum_{n=1}^{\infty} A_n \cos\left(n\theta\right) \\
A_0 &= \alpha - \frac{1}{\pi}\int_0^\pi \frac{dz_c}{dx} d\theta = -\frac{1}{\pi}\int_0^\pi \frac{dz_p}{dx} d\theta \\
A_n &= -\frac{2}{\pi}\int_0^\pi \frac{dz_c}{dx}\cos\left(n\theta\right) d\theta = -\frac{2}{\pi}\int_0^\pi \frac{dz_p}{dx}\cos\left(n\theta\right) d\theta \\
c_l &= 2\pi\left(A_0 + \frac{1}{2}A_1\right) \\
c_{m_{ca}} &= -\frac{\pi}{4}\left(A_1 + A_2\right)
\end{aligned}
$$

  - symmetric problem:

$$
\begin{aligned}
\frac{u^*(\theta)}{U_\infty} &= B_0 - B_0' + \sum_{n=1}^{\infty} B_n \cos\left(n\theta\right) \\
\frac{w\left(\theta\right)}{U_\infty} &= B_0 \tan\left(\frac{\theta}{2}\right) + B_0' \cot\left(\frac{\theta}{2}\right) + \sum_{n=1}^{\infty} B_n \sin\left(n\theta\right)
\end{aligned}
$$

    - ∗ direct problem

$$B_n = \frac{2}{\pi}\int_0^\pi \left[\frac{dz_e}{dx} - B_0 \tan\left(\frac{\theta}{2}\right) - B_0' \cot\left(\frac{\theta}{2}\right)\right] \sin\left(n\theta\right) d\theta$$

    - ∗ inverse problem

$$B_0 - B_0' = \frac{1}{\pi}\int_0^\pi \frac{u\left(\theta\right)}{U_\infty} d\theta$$

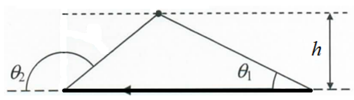
$$B_n = \frac{2}{\pi}\int_0^\pi \frac{u\left(\theta\right)}{U_\infty} cos\left(n\theta\right) d\theta$$

$$B_0 + B_0' + \frac{B_1}{2} = 0$$

3.2 Wing (3D)

- Biot-Savart Law: $\vec{V}(x, y, z) = \dfrac{\lambda}{4\pi}\int_C \dfrac{d\vec{r}' \times \vec{r}}{\|\vec{r}\|^3} d\theta$

  for straight vortex line: $V = \dfrac{\lambda}{4h\pi}\left(\cos\left(\theta_1\right) - \cos\left(\theta_2\right)\right)$



- Lift and drag:

  $L = \dfrac{1}{2}\rho b^2 U_\infty^2 \dfrac{\pi}{2} A_1$ and $D_i = \dfrac{1}{8}\rho b^2 U_\infty^2 \pi \sum_1^\infty n A_n^2$
- Elliptical circulation wing:

  $\alpha_i = \dfrac{\Gamma_o}{2bU_\infty}$

4 Boundary Layer

4.1 2D boundary layer equations

For Stationary, incompressible flow:

- Continuity: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$
- y-momentum: $\frac{\partial p}{\partial y} = 0$
- x-momentum: $\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{dp}{dx} + \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right)$

With boundary conditions:

- Wall: $u(x,0) = v(x,0) = 0$
- B.L. edge: $u(x,\infty) \longrightarrow u_e(x)$ with $\rho_e u_e \frac{du_e}{dx} = -\frac{dp}{dx}$
- On $x = x_0$: $u(x_0, y) = u_0(y)$

4.2 Boundary Layer Characteristic Properties

- Wall Shear Stress: $\tau_w = \mu \left(\frac{\partial u}{\partial y}\right)_w$
- Skin Friction Coefficient: $C_f = \frac{\tau_w}{\frac{1}{2}\rho_e u_e^2}$
- Displacement Thickness: $\delta_1 = \int_0^\delta \left(1 - \frac{\rho u}{\rho_e u_e}\right) dy$
- Momentum Thickness: $\delta_2 = \theta = \int_0^\delta \frac{\rho u}{\rho_e u_e}\left(1 - \frac{u}{u_e}\right) dy$
- Form Factor: $H = \frac{\delta_1}{\theta}$

4.3 Boundary Layer Integral Equations

- Continuity: $\frac{1}{u_e}\frac{d}{dx}\left(u_e\left(\delta - \delta_1\right)\right) = \frac{d\delta}{dx} - \frac{v_\delta}{u_e} = C_E$
- Momentum: $\frac{d\theta}{dx} + \theta\left(\frac{H+2}{u_e}\frac{du_e}{dx}\right) = \frac{C_f}{2}$

4.4 Falkner-Skan Self-Similar Solutions

- Wedge Flows
  - Wall velocity: $u_e(x) = kx^m$
  - Deflection Angle: $\alpha = \frac{\pi\beta}{2}$, with $\beta = \frac{2m}{m+1}$
- Laminar BL Self-Similar Solutions

	β	m	$\frac{Re_\theta}{\sqrt{Re_x}}$	$\frac{Re_{\delta_1}}{\sqrt{Re_x}}$	H	$\frac{C_f}{2}\sqrt{Re_x}$	$\frac{C_f}{2}Re_\theta$	$\frac{\theta Re_\theta}{u_e}\frac{du_e}{dx}$	$Re_\theta \frac{d\theta}{dx}$
	20.000	-1.111	0.3511	0.7303	2.0798	1.2211	0.4288	0.1370	-0.1301
	10.000	-1.250	0.3259	0.6810	2.0895	1.2994	0.4235	0.1328	-0.1195
	2.000	∞	0.0000	0.0000	2.1554	∞	0.3894	0.1065	-0.0533
SP	1.000	1.000	0.2923	0.6479	2.2162	1.2326	0.3603	0.0855	0.0000
	0.500	0.333	0.4290	0.9857	2.2969	0.7575	0.3249	0.0613	0.0613
	0.286	0.167	0.5090	1.2051	2.3678	0.5826	0.2965	0.0432	0.1079
FP	0.000	0.000	0.6641	1.7208	2.5911	0.3321	0.2205	0.0000	0.2205
	-0.040	-0.020	0.6942	1.8440	2.6564	0.2905	0.2017	-0.0095	0.2457
	-0.080	-0.038	0.7279	1.9973	2.7441	0.2451	0.1784	-0.0204	0.2751
	-0.120	-0.057	0.7663	2.2007	2.8718	0.1935	0.1483	-0.0332	0.3102
	-0.160	-0.074	0.8112	2.5082	3.0907	0.1298	0.1054	-0.0488	0.3537
Sep	-0.199	-0.090	0.8681	3.4978	4.0292	0.0000	0.0000	-0.0682	0.4109
	-0.160	-0.074	0.7679	5.1850	6.7520	-0.0854	-0.0656	-0.0437	0.3167
	-0.120	-0.057	0.6369	6.4051	10.0563	-0.0982	-0.0625	-0.0230	0.2143
	-0.080	-0.038	0.4799	7.9023	16.4675	-0.0917	-0.0440	-0.0089	0.1196
	-0.040	-0.020	0.2889	10.3846	35.9436	-0.0677	-0.0196	-0.0016	0.0426

- Flat Plate Boundary Layer (Blasius)
  - Wall Friction Factor: $\tau_w = 0.332\,\rho_e u_e^2\, Re_x^{-1/2}$
  - Displacement Thickness: $\delta_1/x = 1.7208\, Re_x^{-1/2}$
  - Momentum Thickness: $\theta/x = 0.664\, Re_x^{-1/2}$
  - Form Factor: $H = 2.591$
  - Friction force: $F/b = \rho_e u_e^2 \theta(L)$
  - Drag coefficient (one side): $C_D = 2\theta(L)/L$
- 2D Stagnation Point ($u_e = kx$)
  - Wall Friction Factor: $\tau_w = 1.2326\,\rho_e\, kx\,\sqrt{k\nu}$
  - Displacement Thickness: $\delta_1 = 0.6479\,\sqrt{\nu/k}$
  - Momentum Thickness: $\theta = 0.2923\,\sqrt{\nu/k}$
  - Form Factor: $H = 2.2162$
  - $\tau_w = 0$ at stagnation point

4.5 Approximate Integral Method

- Assumptions: $\frac{C_f}{2} = \frac{b}{Re_\theta^{m_0}}$ and H constant
- with

	m_0	b	H
laminar	1	0.2205	2.591
turbulent	1/5	0.0086	1.4

- Solution:

$$\left[\theta u_e^{(H+2)}\right]_{x_1}^{m_0+1} = \left[\theta u_e^{(H+2)}\right]_{x_0}^{m_0+1} + (m_0+1)b\int_{x_0}^{x_1} \frac{u_e^{(m_0+1)(H+2)}}{(u_e/\nu)^{m_0}} dx$$

- Results for flat plate (assuming BL starts at leading edge with $x = 0$):

$p = \frac{k_1}{Re_x^{s_0}} = \frac{k_2}{Re_\theta^{m_0}}$

	Laminar $m_0 = 1$ $s_0 = 1/2$		Turbulent $m_0 = 1/5$ $s_0 = 1/6$	
p	k_1	k_2	k_1	k_2
θ/x	0.664	0.441	0.0221	0.0103
δ_1/x	1.721	1.143	0.0309	0.0144
C_f	0.664	0.441	0.0368	0.0172

4.6 Reynolds Averaged Navier Stokes

- Mass Conservation: $\frac{\partial \overline{u_j}}{\partial x_j} = 0$
- Momentum Conservation:

$$\rho\frac{\partial \overline{u_i}}{\partial t} + \rho\overline{u_j}\frac{\partial \overline{u_i}}{\partial x_j} = -\frac{\partial \overline{p}}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\mu\frac{\partial \overline{u_i}}{\partial x_j} - \rho\,\overline{u_i' u_j'}\right)$$

4.7 2D turbulent boundary layer equations

For Stationary, incompressible flow:

- Continuity: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$
- y-momentum: $\frac{\partial p}{\partial y} = 0$
- x-momentum: $\rho u\frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y} = -\frac{dp}{dx} + \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y} - \rho\,\overline{u'v'}\right)$

With boundary conditions:

- Wall: $u(x,0) = v(x,0) = \overline{u'v'} = 0$
- B.L. edge: $u(x,\infty) \longrightarrow u_e(x)$ and $\overline{u'v'} \to 0$

  with $\rho_e u_e \frac{du_e}{dx} = -\frac{dp}{dx}$
- On $x = x_0$: laminar BL results

Assuming $\overline{u'w'} = \overline{v'w'} = 0$

4.8 Wall units and Mixing length model

- Shear/friction velocity and length: $u_\tau = \sqrt{\frac{\tau_w}{\rho}}$ $\quad l_\tau = \frac{\nu}{u_\tau}$
- Wall variables: $u^+ = \frac{u}{u_\tau}$ $\quad y^+ = \frac{y\,u_\tau}{\nu}$
- Model for Reynold-Stresses: $-\overline{u'v'} = F^2 l^2 \left(\frac{\partial u}{\partial y}\right)^2$
- Experimental Mixing Length: $\frac{l}{\delta} = 0.085\tanh\left(\frac{\chi}{0.085}\frac{y}{\delta}\right)$

  von Karman constant $\chi = 0.41$
- Damping Function: $F = 1 - exp\left(-\frac{y^+}{A^+}\right)$; $A^+ = 26$

4.9 Miscellaneous

- Squire & Young Formula:

$$C_D = 2\frac{\theta\infty}{L} = 2\left[\frac{\theta_{te_u}}{L}\left(\frac{u_{te_u}}{U_\infty}\right)^{\frac{H_{te_u}+5}{2}} + \frac{\theta_{te_u}}{L}\left(\frac{u_{te_l}}{U_\infty}\right)^{\frac{H_{te_l}+5}{2}}\right]$$

- Squire & Young Formula (symmetric airfoil, $\alpha = 0$):

$$C_D = 2\frac{\theta\infty}{L} = 4\frac{\theta_{te}}{L}\left(\frac{u_{te}}{U_\infty}\right)^{\frac{H_{te}+5}{2}}$$